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MECCANICA DEL CONTINUO

Finite difference in unidimensional linear dynamics with time dependent boundary location

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Riassunto. *L'equazione delle corde vibranti è l'equazione di campo fondamentale per i problemi in esame. Le onde che si propagano, che costituiscono la base per ogni soluzione di tale equazione, hanno proprietà di riflessione, su contorni aventi ubicazioni dipendenti dal tempo, che l'A. ha studiato in precedenza e che sono qui richiamate.*

Si propone una analisi teorica che consenta l'individuazione delle due onde propagantisi, quando siano noti, ad un certo tempo, gli andamenti degli spostamenti e delle velocità.

Sulla base di tale corpo di conoscenze si propone e si applica ad alcuni interessanti problemi un approccio discretizzato alla dinamica lineare unidimensionale con ubicazione degli estremi dipendente dal tempo.

Abstract. *The vibrating string equation is the fundamental field equation of the proposed problem. The travelling waves, which constitutes the base for each solution of such equation, have properties of reflection on boundaries having time dependent location, previously studied by the A. and here recalled.*

A theoretical analysis which permits the determination of the two travelling waves if displacements and their speeds are known at each time.

On the basis of such body of knowledges a discretised approach of the unidimensional linear dynamics with time dependent boundary location is proposed and applied to several interesting problems.

1. Introduction

In the unidimensional dynamic problems in non dispersive media, as the extensional wave propagation in an unidimensional body, the vibrating string equation holds as field equation:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}, \qquad 1.1)$$

where x = space coordinate, t = time coordinate, c = propagation speed of longitudinal phenomena ($c = \sqrt{E/\rho}$, E being the elastic modulus and ρ the density of the constitutive material), and where ψ can be identified as the displacement s, the material speed u, the longitudinal local stress σ or other quantities obtainable by means of derivations starting from ψ.
As it is well know, eq. 1) admits the solution

$$\psi = \psi_1\left(t - \frac{x}{c}\right) + \psi_2\left(t + \frac{x}{c}\right) \qquad 1.2)$$

composed of two travelling waves: ψ_1 in the positive x direction and ψ_2 in the negative x direction. The function ψ_1 and ψ_2 can be of whichever shape, as generated by the boundary conditions.

In the case of time dependent boundary location, we have the problem of the reflections of such two waves on the boundary themselves. The A. demonstrated, [1], that the reflections of the waves are ruled by a foundamental law concerning the material speeds u_i (or the local stress σ_i) ($i = 1,2$). If v is the speed of the change of the boundary ($u = 0$) location, positive if directed as the propagation wave, it results:

$$u_j = -u_i \frac{c - v}{c + v}, \qquad (i \neq j). \qquad 1.3)$$

In other words, in the case of a boundary condition $u = 0$, whose location has a speed v, an incident wave, having a speed u_1 generates a reflected wave having a speed $u_2 = -u_1 \dfrac{c - v}{c + v}$.

A tipical problem one has interest to resolve, is the following: if an initial status is defined by means of a displacement function $s(x,t)$, and a speed function $u = \dot{s}(x,t_s)$, both defined in an interval from x_A to x_B at a time t_0,determine the evolution of the dynamic phenomenon, where the boundary conditions $(\dot{s} = 0)$ have locations depending on t (and obviously coherent with x_A and x_B at t_0).

Other problems can be referred to introducing external actions or boundary conditions of different natures.

If the afore mentioned typical problem is taken into account, we can consider the possibility of facing it by means of a discretised (i.e. approssimated) finite difference method, that could be utilized in automatic means. The afore mentioned foundamental law that rules the reflection of waves can be taken as the correct behaviour and a paragon for the discretised calculation.

2. A theoretical analysis supporting a finite difference approach

Each dynamic phenomenon $s(x,t)$ can be considered as the sum of two travelling waves $s_1(x-ct)$ and $s_2(x+ct)$. If $s(x,t_0)$ and $u(x,t_0)$ are known, the first step is to calculate the proper functions s_1 and s_2, we can indicate respectively with g and f. Let's considere the solution into the form 1.2). A local value (in x) of the speed for $t=0$, given by

$$u(x,0)=\left[\frac{\partial}{\partial t}u(x,t)\right]_{t=0}, \qquad 2.1)$$

corresponds to each $s(x,t)$. Equation 1.2) can be put in the form:

$$s(x,t)=f(x+ct)+g(x-ct) \qquad 2.2)$$

and we have:

$$f(x)+g(x)=s(x,0), \qquad 2.3)$$

$$\left[\frac{\partial}{\partial t}f(x+ct)\right]_{t=0}-\left[\frac{\partial}{\partial t}g(x-ct)\right]_{t=0}=\left[\frac{\partial s}{\partial t}(x,t)\right]_{t=0}, \qquad 2.4)$$

where

$$\left[\frac{\partial}{\partial t}f(x+ct)\right]_{t=0}=c\left[\frac{d}{d(x+ct)}f(x+ct)\right]_{t=0}.$$

An analogous expression holds for $g(x)$. Let us consider a dt and the correspondent $dx=cdt$. Because of the propagational nature of g and f, the following equation holds:

$$f[(x+cdt)+c(t-dt)]+g[(x-cdt)-c(t-dt)]=s(x,t), \qquad 2.5)$$

giving $s(x,t)$ as a sum of two contributions evaluated at the time $t-dt$ in the proper positions. Therefore we can write:

$$s(x,dt)=f(x+cdt,0)+g(x-cdt,0),$$

and

$$s(x,-dt)=f(x-cdt,0)+g(x+cdt,0).$$

On the other hand, it is also:

$$u(x,0) = [s(x,dt) - s(x,-dt)]\frac{1}{2dt}.$$

Therefore we have:

$$u(x,0) = \frac{[f(x+cdt) + g(x-cdt)] - [f(x-cdt) + g(x+cdt)]}{2dt}. \qquad 2.6)$$

We can arrive to the same expression in the following manner. Let's evaluate:

$$\dot{f}(x) = \frac{f(x+cdt) - f(x-cdt)}{2dt}, \qquad 2.7)$$

$$\dot{g}(x) = \frac{g(x-cdt) - g(x+cdt)}{2dt}. \qquad 2.8)$$

Introducing 2.7) and 2.8) into eq. 2.4), we obtain:

$$2dt\dot{u}(x,0) = f(x+cdt) - f(x-cdt) + g(x-cdt) - g(x+cdt), \qquad 2.9)$$

and, operating, we can realise that:

$$s(x+cdt,0) = f(x+cdt) + g(x-cdt),$$

$$s(x-cdt,0) = f(x-cdt) + g(x+cdt)$$

and, as a final result:

$$2dt\dot{u}(x,0) = s(x+cdt,0) - s(x-cdt,0). \qquad 2.10)$$

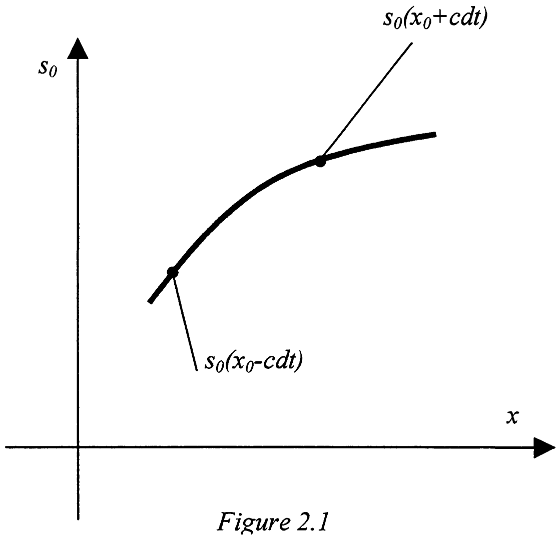

Figure 2.1

If $s(x,t^*)$ and $u(x,t^*)$ are given, where t^* is an assigned time value, it is possible to determine the two travelling waves $f(x+ct)$ and $g(x-ct)$, which the phenomenon can be decomposed in.

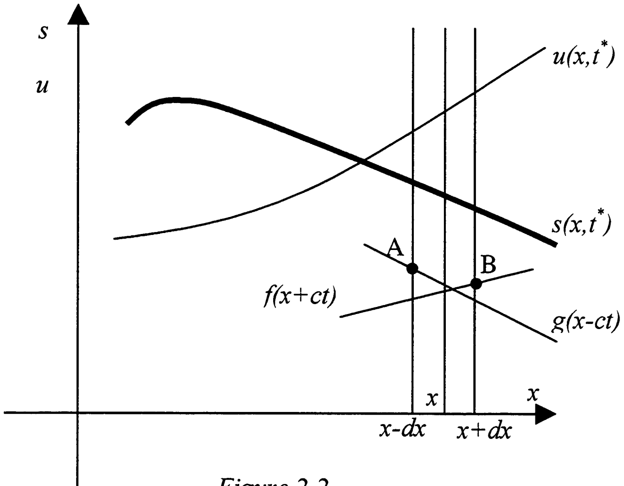

Figure 2.2

We can write:

$$s(x,t)=f(x+ct)+g(x-ct)=f(q)+g(p), \qquad 2.11)$$

where $q=x+ct$, $p=x-ct$. Therefore:

$$u(x,t)=c\frac{df}{dq}-c\frac{dg}{dp}, \qquad 2.12)$$

and in particular, as time t^* is concerned:

$$u(x,t^*)=c\left(\frac{df}{dq}\right)_{x,t^*}-c\left(\frac{dg}{dp}\right)_{x,t^*}. \qquad 2.12')$$

From eq. 2.11) we have:

$$s(x,t^*)=f\left[x+dx+c\left(t^*-\frac{dx}{c}\right)\right]+g\left[x-dx-c\left(t^*-\frac{dx}{c}\right)\right], \qquad 2.13)$$

where the displacement in x at t^* is obtained from values of g and f in appropriately changed position and time. Similary we have:

$$s\left(x,t^*+\frac{dx}{c}\right)=f\left[x+dx+ct^*\right]+g\left[x-dx-ct^*\right]. \qquad 2.14)$$

Putting now the speed $u(x,t)$ under the form:

$$u(x,t)=\left[s\left(x,t^{*}+\frac{dx}{c}\right)-s\left(x,t^{*}\right)\right]\frac{c}{dx},$$

we obtain:

$$u\left(x,t^{*}\right)=\frac{c}{dt}\left\{f\left(x+dx+ct^{*}\right)-f\left(x+ct^{*}\right)+g\left(x-dx-ct^{*}\right)-g\left(x-ct^{*}\right)\right\}$$

or

$$u\left(x,t^{*}\right)=c\left\{\left(\frac{\partial f}{\partial q}\right)_{x,t^{*}}-\left(\frac{\partial g}{\partial p}\right)_{x,t^{*}}\right\}, \qquad 2.12'')$$

which is another form for the meaning of eq. 2.12'). Deriving for respect to x eq. 2.11) and putting eq. 2.12'') in another form, we obtain:

$$\frac{\partial s}{\partial x}(x,t)=\frac{\partial f}{\partial q}(x+ct)+\frac{\partial g}{\partial p}(x-ct), \qquad 2.15)$$

$$\frac{1}{c}\frac{\partial s}{\partial t}(x,t)=\frac{\partial f}{\partial q}(x+ct)-\frac{\partial g}{\partial p}(x-ct) \qquad 2.16)$$

or, in other words:

$$s_x(x,t)=f_q(x+ct)+g_p(x-ct), \qquad 2.15')$$

$$\frac{1}{c}u(x,t)=f_q(x+ct)-g_p(x-ct), \qquad 2.16')$$

If we put $s=s\left(x,t^{*}\right)$, $f=f\left(x+ct^{*}\right)$, $g=g\left(x-ct^{*}\right)$, $f_x=\left(\frac{\partial f}{\partial q}\right)_{x,t^{*}}$, $g_x=\left(\frac{\partial f}{\partial p}\right)_{x,t^{*}}$, from 2.11) and 2.12), we can obtain:

$$s=f+g \text{ and } g=s-f,$$

$$\frac{u}{c}=\frac{\partial f}{\partial q}-\frac{\partial g}{\partial p}=f_x-s_x+f_x=2f_x-s_x.$$

Therefore we have:

$$f_x=\frac{s_x+u/c}{2}, \qquad 2.13a)$$

$$g_x=\frac{s_x-u/c}{2}, \qquad 2.13b)$$

Eq.s 13a) and 13b) can be integrated, as follows:

$$f=\frac{s}{2}+\int\frac{1}{2}\frac{\dot{u}}{c}dx+c_1,$$

$$g = \frac{s}{2} - \int \frac{1}{2}\frac{\dot{u}}{c}\,dx + c_2 ,$$

where it results $c_2 = -c_1$, because of eq. 2.11).

On the other hand, $c_2 = -c_1$ is arbitrary but indifferent, because $f = c_1$, $g = -c_1$ generate $s = 0$ and $u = 0$. Therefore it is possible to choose $c_1 = 0$:

$$f\left(x + ct^*\right) = \frac{s}{2} + \int \frac{u}{2c}\,dx , \qquad 2.16a)$$

$$g\left(x - ct^*\right) = \frac{s}{2} - \int \frac{u}{2c}\,dx . \qquad 2.16b)$$

Eq.s 2.16a) and 2.16b) allow us to obtain f and g from s and u. When $s\left(x,t^*\right)$ and $\dot{u}\left(x,t^*\right)$ are known, it is possible to calculate $\int u dx$, starting from a value $\bar{x}$ of x, and to determine f and g by means of 2.16a) and 2.16b). Such functions $f(x+ct)$ and $g(x-ct)$ are travelling waves.

3. A finite difference approach

The considered problem is defined in $x_A(t) \longmapsto x_B(t)$ see fig. 3.1. The first increments of x_A and x_B are:

$x_A(t_0 + \Delta t) = x_A(t_0) - v_A \Delta t$ and $x_B(t_0 + \Delta t) = x_B(t_0) + v_B \Delta t$,

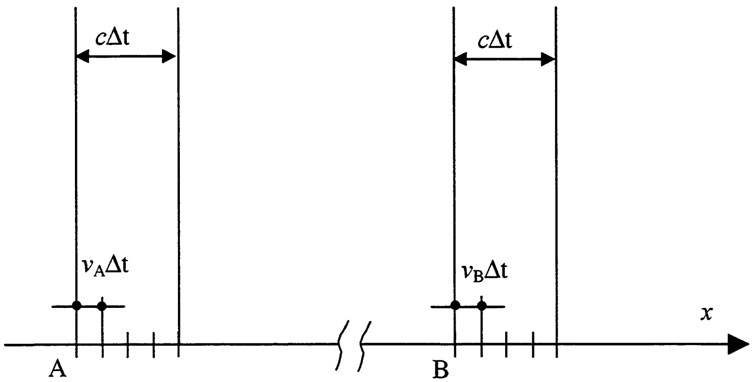

Figure 3.1- Interval of definition and its time

where v_A and v_B are positive in the positive direction of x and where Δt is the discrete increment of t, we are taking as step in a finite difference approach. In other words $v_A \Delta t$ and $v_B \Delta t$ are the displacements of the locations A and B of the boundary conditions.

Let us suppose now that $|c| = k_A |v_A|$ and $|c| = k_B |v_B|$, where k_A and k_B are integer numbers and c is the propagation speed of dynamic fenomena along x. Real problems are characterised by very high k_A and k_B values: to consider only integer values for them has the scope of reducing our considerations to such particular cases, requiring simple algorithm and formulas. Another simplification we shall adopt is to take into account values of v_A and v_B of the following type:

$$v_A = \begin{cases} +v \\ 0, \\ -v \end{cases} \quad v_B = \begin{cases} +v \\ 0 \\ -v \end{cases} \qquad 3.1)$$

and to consider $v_A(t)$ and $v_B(t)$ piece-wise curves having their discontinuities in corrispondence of multiples of Δt, see fig. 3.2.

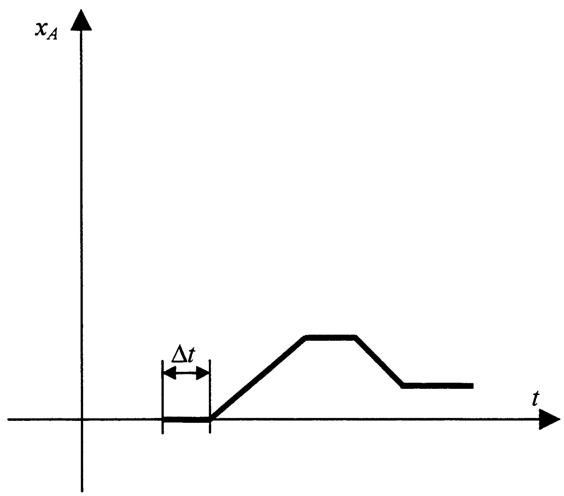

Figure 3.2- Example of a picewise $x_A=x_A(t)$ having discontinuities at the ends of Δt interval and dx_A/dt equal to v, or 0, or $-v$

The aim of this work is to demonstrate the possibility of approaching a vibrating chord problem, where the boundary conditions have locations depending on time, by means of a finite difference approach, coherente with a theoretical formulation. To such a purpose the considered problem is sufficient. The determination of algorithms suitable in more complex problems is beyond the scope of the present work.

At the time t_0 A and B had locations $x_A(t_0)$ and $x_B(t_0)$. Let the interval from $x_A(t_0)$ to $x_B(t_0)$ be a multiple of $v\Delta t$.

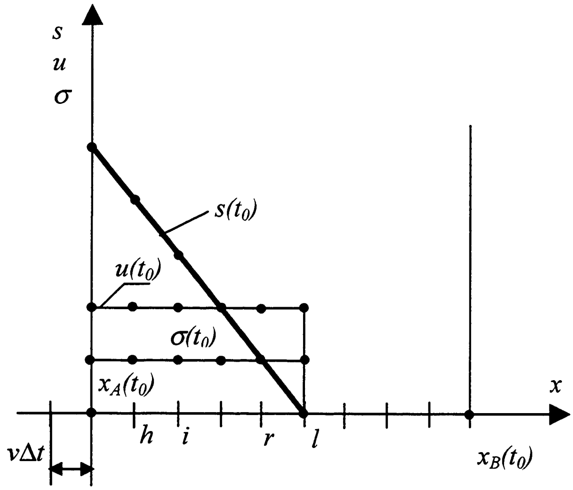

Figure 3.3- Discretized data on $s(t_0)$, $u(t_0)$ and $\sigma(t_0)$ for a simple problem

The typical dynamic problem to be considered is the determination of the evolution for $t > t_0$ of an initial $(t = t_0)$ state where $s(t_0)$, $u(t_0)$ and $\sigma(t_0)$ are given. In order the problem can result well posed, such initial data have to be obviously coherente. It is necessary first to discretise such data by means of the values corresponding to the dots indicated in fig. 3.3). The determination of the (discretized) travelling function $f(t_0)$ and $g(t_0)$, can be obtained as follows:

$$g_r(t_0) = \frac{s_r(t_0)}{2} - \sum_{i=h}^{r} \frac{u_i(t_0)}{2c} v\Delta t + c_1 , \qquad 3.2)$$

where h is an arbitrary value of r and
i is a variable value $(h \le i \le r)$,

$$f_r(t_0) = \frac{s_r(t_0)}{2} + \sum_{i=h}^{r} \frac{u_i(t_0)}{2c} v\Delta t - c_1 \qquad 3.3)$$

and where c_1 is a constant to be determined on te basis of σ's values (for instance at a boundary).

In the case of travelling wave having $\sigma = const$, as follows, [1]:

$$s(x,t_0)\begin{cases} = a - bx & (x \le l), \\ = 0 & (x > l), \end{cases}$$
$$u(x,t_0)\begin{cases} = bc & (x \le l), \\ = 0 & (x > l), \end{cases} \qquad 3.4)$$
$$\sigma(x,t)\begin{cases} = Eb & (x \le l), \\ = 0 & (x > l), \end{cases}$$

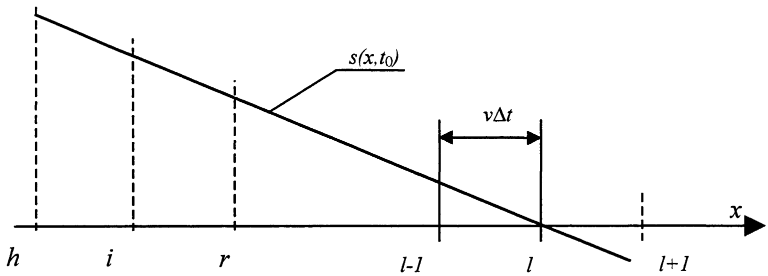

Figure 3.4- A *σ=const* travelling wave

the discretization gives:

$$s_i(t_0) = (l-h)bv\Delta t - (i-h)bv\Delta t = (l-i)bv\Delta t,$$
$$u_i(t_0) = bc,$$

with $\sigma_i(t_0) = Eb$.

Therefore:

$$g_r(t_0)=\frac{s_r(t_0)}{2}-\sum_{i=h}^{r}\frac{b}{2}v\Delta t-c_1,$$

$$f_r(t_0)=\frac{s_r(t_0)}{2}+\sum_{i=h}^{r}\frac{b}{2}v\Delta t+c_1.$$

Operating we have:

$$\begin{cases} g_r(t_0)=\dfrac{(l-r)bv\Delta t}{2}-\dfrac{bv\Delta t}{2}(r-h)-c_1=(l-2r+h)\dfrac{bv\Delta t}{2}-c_1, \\ f_r(t_0)=\dfrac{(l-r)bv\Delta t}{2}+\dfrac{bv\Delta t}{2}(r-h)+c_1=(l-h)\dfrac{bv\Delta t}{2}+c_1. \end{cases}$$

A choise of $c_1=-(l-h)\dfrac{bv\Delta t}{2}$ allows us to have $f_r(t_0)=0$ and

$$g_r(t_0)\begin{cases} =(l-r)bv\Delta t & (h<r<l), \\ =0 & (r>l). \end{cases} \qquad 3.5)$$

This expression 3.5) coincide with the proposed 3.4). $s(x,t_0)=a-bx$, if we pose $a=lbv\Delta t$ and $x=rv\Delta t$.

The proposed motion 3.4), can be interpreted by 3.5), $g_r(t_0)$ being a displacement function travelling rightward at the speed c. During the interval Δt, g has a traslation of $kv\Delta t$. If the location not time dependent of the boundary x_B is greater of $(l+k)v\Delta t$, we have:

$$g_r(t_0+\Delta t)=(l-r+k)bv\Delta t \qquad (h<r<l+k),$$

$$g_r(t_0+\Delta t)=0 \qquad (r>l+r).$$

On the other hand, if $x_B<(l+k-1)v\Delta t$, for instance $x_B=(l+1)v\Delta t$, we have:

$$s(l+1,t_0+\Delta t)=0,$$

but it is:

$$g_{l+1}(t_0+\Delta t)=(k-1)bv\Delta t,$$

hence there exists, see fig. 3.5:

$$f_{l+1}(t_0+\Delta t)=-(k-1)bv\Delta t.$$

Such a leftward running function obviously initiated at the time $t_0 + \frac{\Delta t}{k}$ with a zero value, and runned leftward during the time $\frac{(k-1)}{k}\Delta t$. The boundary at $l+1$ introduces a reflected wave f whose effect is the zeroing

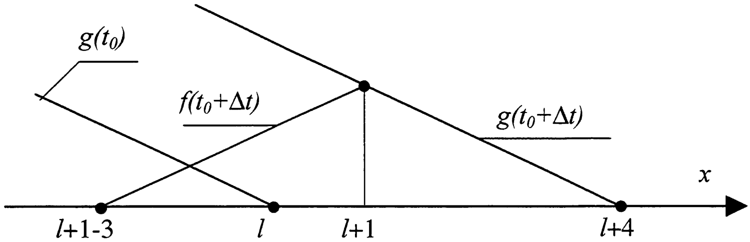

Figure 3.5- Impact of a *σ=const* travelling wave with a fixed boundary
Note . This figure concerns the case *k=4*

of $s(l+1, t_0 + \Delta t)$. It is very easy to verify that, if the boundary rests at $l+1$, after another Δt, at $t_0 + 2\Delta t$,

$$g_{l+1}(t_0 + 2\Delta t) = (2k-1)v\Delta t$$

with the necessity of a function:

$$f_{l+1}(t_0 + 2\Delta t) = -(2k-1)v\Delta t,$$

and so on.

As an example of time dependent boundary condition we can consider the case

$$s(l, t_0) = 0,$$
$$s(l+1, t_0 + \Delta t) = 0,$$
$$s(l+2, t_0 + 2\Delta t) = 0,$$

where, during the period from $t_0 + \Delta t$ to $t_0 + 2\Delta t$, the boundary condition translates from $l+1$ to $l+2$ and during the period from t_0 to $t_0 + \Delta t$ from l to $l+1$.

At the time t_0 in l starts the f whose zero value reaches in Δt the position *l-k*.

At the time $t_0 + \Delta t$, we have

$$f_{l+1}(t_0 + \Delta t) = -(k-1)bv\Delta\tau .$$

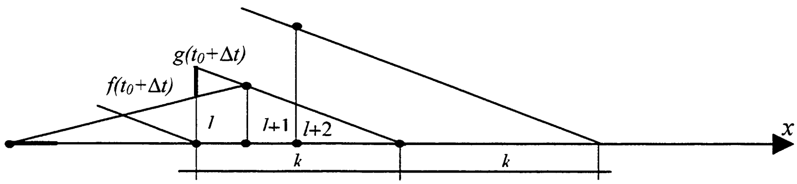

Figure 3.6- Impact of a s=const travelling wave on time dependent boundary

The slope of f results

$$-\frac{(k-1)bv\Delta\tau}{(k+1)v\Delta t} = -b\frac{k-1}{k+1} .$$

During another Δt the boundary translates in $l+2$, the zero value of f runs to the point $(l-2k+1)$ whose distance from $l+2$ is $2(k+1)$ and it is

$$f_{l+2}(t_0 + 2\Delta t) = -2(k-1)bv\Delta t .$$

Therefore the slope of f is an invariant if v is an invariant.

4. Examples of various boundary conditions

4.1. *Condition of zero displacement running toward the outside of the definition interval*

Let us propose several examples of boundary conditions, analysed in simplified cases, such as:

i) $k = \frac{c}{v}$ assumed to be an integer number, *ii)* stress applied at the boundary having unitary values and so on. The generalization of such examples, their combination in a superposition of effects, the study of cases here non considered, are left to the reader. As an example of time dipendent boundary condition we can consider the case, (fig 4.1):

$$s(l,t_0)=0\,,$$
$$s(l+1,t_0+\Delta t)=0\,,$$
$$s(l+2,t_0+2\Delta t)=0\,,$$

where, during the period from $t_0+\Delta t$ to $t_0+2\Delta t$, the boundary condition translates from $l+1$ to $l+2$ and, during the period from t_0 to $t_0+\Delta t$ from l to $l+1$. Obviously a travelling g is acting. At the time t_0 in l start the f whose zero value reaches in Δt the position $l-k$.

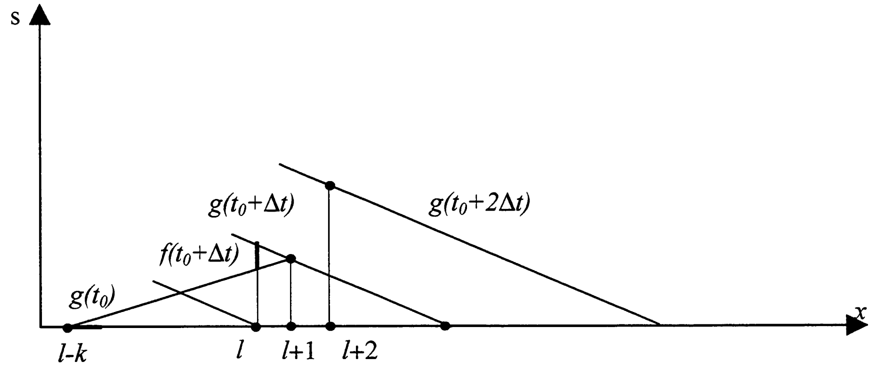

Figure 4.1- Time dependent boundary condition

At the time $t_0+\Delta t$, we have

$$f_{l+1}(t_0+\Delta t)=-(k-1)bv\Delta t\,.$$

The slope of f results

$$\frac{(k-1)bv\Delta t}{(k+1)v\Delta t}=b\frac{k-1}{k+1}\,.$$

During another Δt the boundary translates in $l+2$, the zero value of f runs to the point $(l-2k)$ whose distance from $l+2$ is $2(k+1)$ and it is

$$f_{l+2}(t_0+2\Delta t)=-2(k-1)bv\Delta t\,.$$

Therefore the slope of f is an invariant if v is an invariant.

4.2. *Condition of zero displacement running toward the inside of the definition interval*

Let us now consider the case (fig. 4.2):

$$s(l,t_0)=0\,,$$
$$s(l-1,t_0+\Delta t)=0\,,$$
$$s(l-2,t_0+2\Delta t)=0\,,$$

where, during the period from $t_0+\Delta t$ to $t_0+2\Delta t$, the boundary condition translates from $l-1$ to $l-2$ and, during the period from t_0 to $t_0+\Delta t$, from l to $l-1$. Obviously a travelling g is acting. At the time t_0 in l start the f whose zero value reaches in Δt the position $l-k$. At the time $t_0+\Delta t$, we have:

$$f_{l-1}(t_0+\Delta t)=-(k+1)bv\Delta t\,.$$

The slope of f results

$$\frac{(k+1)bv\Delta t}{(k-1)v\Delta t}=b\frac{k+1}{k-1}\,.$$

During another Δt the boundary translates in $l-2$, the zero value of f runs to the point $(l-2k)$, whose distance from $l-2$ is $2(k-1)$ and it is:

$$f_{l-r}(t_0+2\Delta t)=-2(k+1)bv\Delta t\,.$$

Therefore the slope of f is an invariant if v is an invariant.

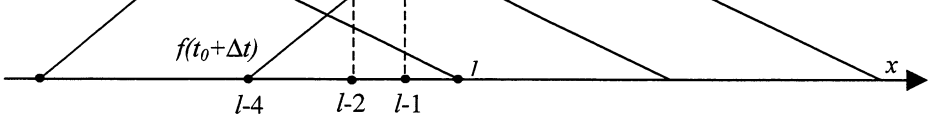

Figure 4.2- Condition of zero displacement running toward the inside of the definition interval

4.3. *Condition of imposed stress running toward the inside of the definition interval and no travelling functions are acting*

Let us now consider another problem, due to a condition at the boundary B, whose location translates at a speed $v = -1$, and where:

$$\sigma = 1(x = x_b).$$

To notice that in the proposed problem only a displacement function *f(x+ct)* will be present. Therefore between u and σ the following relation holds:

$$\sigma = \frac{E}{c} u.$$

At the time t_0 the boundary is in l, fig. 4.3); at the time $t_0 + \Delta t$ the boundary is in $l - 1$, and so on. At t_0 in l start a wave, that, if $k = 4$, during Δt reaches $l - 4$, while the condition $\sigma = 1$ reaches $l - 1$.

4.4. *Condition of imposed stress running toward the outside of the definition interval and no travelling functions are acting*

The problem is analogous to the previous one, but with $v = +1$, and

$$x_b = l + (t_0 - t).$$

The stress condition is always $\sigma = 1(x = x_b)$. At the time t_0 the boundary is in l, (fig. 4.4); at the time $t_0 + \Delta t$ the boundary is in $l + 1$, and so on. At t_0 in l start a wave, that if $k = 4$ during Δt reaches $l - 4$, while the condition $\sigma = 1$ reaches $l + 1$.

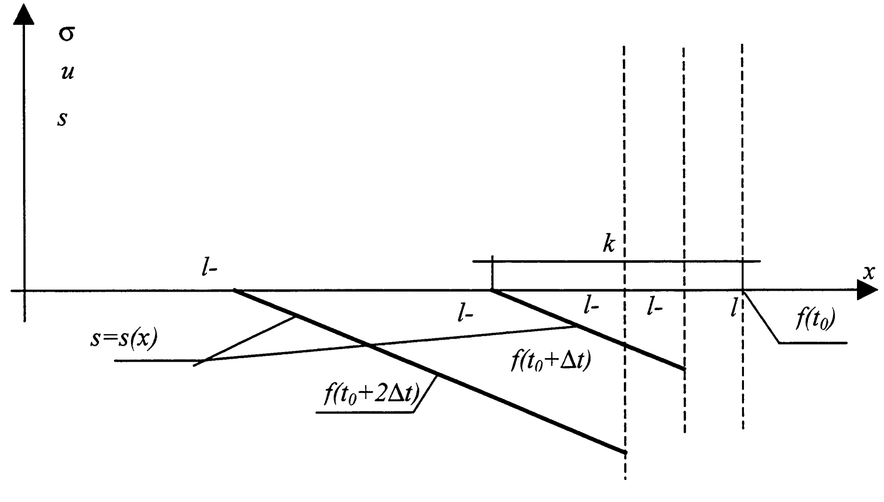

Figure 4.3. Imposed stress running toward the inside

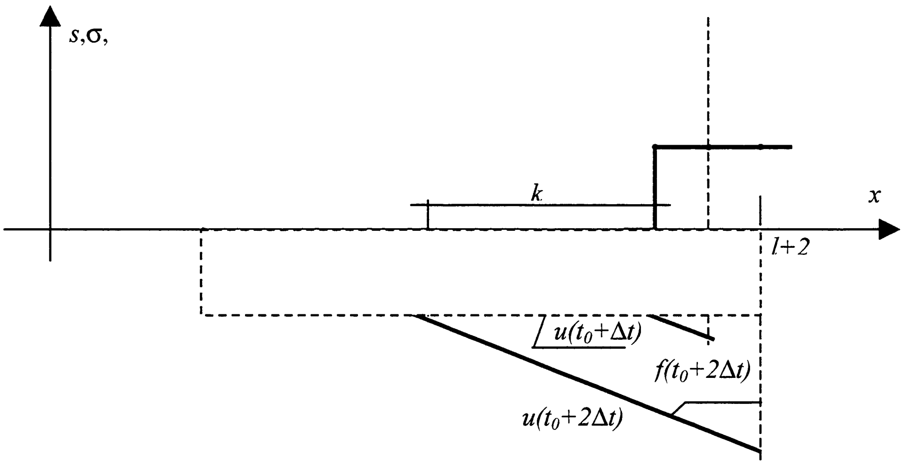

Figure 4.4. Imposed stress running toward the outside

APPENDIX A

If $s = s(x,t^*) = a - bx$, and $u = u(x,t^*) = bc$, we have:

$$f_x = \frac{s_x}{2} + \frac{1}{2c}u, \qquad \text{A1)}$$

$$g_x = \frac{s_x}{2} - \frac{1}{2c}u. \qquad \text{A2)}$$

From A1) and A2) one obtains:

$$f = \frac{s}{2} + \frac{1}{2c}\int u dx, \qquad \text{A3)}$$

$$g = \frac{s}{2} - \frac{1}{2c}\int u dx. \qquad \text{A4)}$$

Substituting into A3) and A4) the initial data, we obtain:

$$f = \frac{1}{2}(a - bx) + \frac{1}{2c}bcx + c_1 = \tfrac{1}{2}a + c_1,$$
$$g = \frac{1}{2}(a - bx) - \frac{1}{2c}bcx - c_1 = \tfrac{1}{2}a - bx - c_1.$$

With an indifferent value $c_1 = -\frac{1}{2}a$, the assumed situation is retrouved.

In fact

$$f = 0,$$
$$g = a - b[x - c(t - t^*)],$$

and such travelling weaves give:

$$s[x,(t - t^*)] = a - b[x - c(t - t^*)],$$
$$u[x,(t - t^*)] = bc.$$

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