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# A Finite Difference Approach to Linear Continuum Dynamics as Wave Propagation

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**Abstract.** *Any elastic dynamic phenomenon in continuous, non-bounded, homogeneous and isotropic media without external field forces has the nature of spherical wave propagations. There are only two kinds of spherical waves with different propagation speeds. At each time is always possible to separate the motion in two components, propagating respectively in the two kinds of waves. The motion at a point at a given time depends on the motions of the two kinds of waves at a previous time on the surfaces of two conventional spheres having as radii the distances travelled respectively by the two spherical waves. One can consider the influence of the sphere on its centre. A discretized solution by means of a finite difference approach is proposed, where a cubic network allows us to consider each point as the centre of various spheres having as radii multiples of the cube diagonal. Obviously the boundary of the volume interested with the dynamic phenomena, in order to have consistency of the approach, must be composed by points of the network, having in such a case the meaning of representative points of discretized areas. The presence of the boundary makes non-applicable, or in any case non-sufficient, in these regions the property of the wave propagtions. Algorithms are presented in a shape including both field relations and boundary ones, where time variable restraint locations are included.*

**Key words.** *continuous dynamics; spherical propagation; discretization; finite differences; irrotational; solenoidal; boundary conditions.*

**Riassunto.** *Ogni fenomeno dinamico elastico in un mezzo continuo, non limitato, omogeneo e isotropo in assenza di forze esterne ha la natura di propagazioni per onde sferiche. Ci sono solo due tipi di onde*

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*sferiche aventi differenti velocità di propagazione. In ogni istante è sempre possibile suddividere lo stato di moto in due componenti, propagantisi rispettivamente nei due tipi di onde. Il moto in un punto a un certo tempo dipende dai moti dei due tipi di onde a un tempo precedente su due superfici sferiche aventi raggi pari alle distanze coperte dalle due onde sferiche. È possibile considerare l'influenza di una superficie sferica sul proprio centro. Una soluzione discretizzata per mezzo di differenze finite viene proposta: un reticolo cubico consente di considerare ogni suo punto quale centro di sfere aventi raggi multipli della diagonale del cubo. Ovviamente il contorno del volume interessato dal fenomeno dinamico, affinché l'approccio sia consistente, deve essere composto da punti del reticolo, aventi in questo caso il significato di punti rappresentativi di volumi discretizzati. La presenza del contorno rende non applicabile, o in ogni caso non sufficiente, in tali regioni la proprietà delle propagazioni ondose. Vengono proposti algoritmi in una forma includente sia le equazioni di campo sia le condizioni al contorno, ivi comprese condizioni aventi ubicazione variabile nel tempo.*

## 1. Introduction

The aim of this paper is to give a foundamental contribution to the solution of dynamic problems in continuous, homogeneous and isotropic media on the basis of the wave propagation nature of such phenomena, in particular with the scope of solving three dimensional dynamic problems with time variable restraint location. In previous papers the A. considered the behaviour of a bar (unidimensional elastic non-dispersive system) having time dependent location of the boundary conditions. More exactly, the case was considered of a time moving section which separates the bar into a region at rest and a region where elastic waves can run along both the two opposite directions. In particular "the reflection" of a wave running in the same direction of the translation speed of the moving section was analysed; the conclusion was that, if $u_1$ is the speed of the disturbancy (running at the sound speed $c$) and $v$ is the speed of the location of the restraint section, the speed of the reflected wave is, [2]:

$$u_2 = - u_1 \frac{c-v}{c+v},$$

where one can see analogies with the Doppler effect. The analysis was extended, [5]:

- to the negative values of $v$ (the restraint section moves in opposite direction with respect to the propagation speed),
- to the case of time depending amplitude of the travelling wave,
- to the case of time depending speed of the boundary condition,
- to the case where the propagation speed is depending on the longitudinal position.

The one-dimensional system analysis has evidenced the travelling wave nature of the dynamic behaviour as the key aspect for the solution of time variable restraint problems. As three dimensional problems are concerned, this paper is an attempt of solving the problem of time variable restraint location by means of well known properties of the dynamic phenomena. Mathematical physics shows the nature of wave propagations of every elastic dynamic phenomenon in continuous, homogeneous and isotropic media, without external force field, analysed by means of linear equations in three-dimensional space. On the whole, involved waves are of two kinds, with two propagation speeds, respectively; then we have two characteristic propagations. In particular, this point of view allows to mean the characteristics of motion, at a given time, at each point, in function of the only characteristics at a previous time, on a spherical surface with a radius depending on the wave kind and with its centre placed in the concerned point. Therefore, “two spheres of influence” are singled out for each point. In other words, mathematical physics introduces theories allowing to follow the time evolution of dynamic phenomena as wave propagation phenomena.

## 2. Linear Continuum Dynamics as Wave Propagations

Persico, [1], has elegantly treated the problem, and we shall follow his analysis in order to present a basis for a clear understanding of the discretization proposed in the present paper. Except for external forces, the linearized equations of dynamics of elastic, homogeneous and isotropic bodies, are summarized in the vectorial field equation[1] [1]:

$$k \frac{\partial^2 \mathbf{s}}{\partial t^2} - (\lambda + \mu) \text{ grad div } \mathbf{s} - \mu \Delta \mathbf{s} = 0, \tag{1}$$

where $\mathbf{s}$ is the displacement, $k$ is the density and $\lambda$, $\mu$ are the Lamé’s constants.

1 When applied to a function (or vector) function $\Delta$ is the Lapace’s operator.

Following such a field equation, every dynamic phenomenon can be considered as the propagation of phenomena of two foundamental kinds. It comes from the chance to make[2] always the decomposition:

$$\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2\,, \quad \text{where} \quad \text{rot}\,\mathbf{s}_1 = 0 \quad \text{and} \quad \text{div}\,\mathbf{s}_2 = 0,$$

where $\mathbf{s}_1$ is an irrotational displacement and $\mathbf{s}_2$ is a solenoidal one.
First, the case $\mathbf{s}_1$, where there exists a scalar $\varphi$ such that $\mathbf{s}_1 = \text{grad}\,\varphi$, gives:

$$\text{grad div}\,\mathbf{s}_1 = \text{grad div grad}\,\varphi = \Delta\mathbf{s}_1.$$

Thus, Eq. 1 becomes [1]:

$$k\frac{\partial^2\mathbf{s}_1}{\partial t^2} - (\lambda+2\mu)\,\Delta\mathbf{s}_1 = 0 \qquad (1')$$

or:

$$\frac{\partial^2\mathbf{s}_1}{\partial t^2} - a^2\Delta\mathbf{s}_1 = 0 \left(a = \sqrt{\frac{\lambda + 2\mu}{k}}\right), \quad (1'\ \text{rep}),$$

that is a D'Alembert equation, satisfied by every propagation of wave surfaces, travelling on perpendicularly (in normal direction to the surface itself) with speed $a$.
In the case of $\mathbf{s}_2$, where $\text{div}\,\mathbf{s}_2 = 0$, [1]:

$$k\frac{\partial^2\mathbf{s}_2}{\partial t^2} - \mu\Delta\mathbf{s}_2 = 0 \qquad (1'')$$

or:

$$\frac{\partial^2\mathbf{s}_2}{\partial t^2} - b^2\Delta\mathbf{s}_2 = 0 \left(b = \sqrt{\frac{\mu}{k}}\right), \qquad (1''\ \text{rep})$$

that is, again, a D'Alembert equation, for which wave surfaces propagate with speed $b$. If it is considered the case of a displacement $\mathbf{s}$, sum of an irrotational displacement $\mathbf{s}_1$ , - i.e. a particular solution of Eq. 1' - , and a solenoidal one $\mathbf{s}_2$ , - i.e. a particular solution of Eq. 1'' - , it results to be a particular solution of Eq. 1 because of its linearity. So, the nature of dynamic phenomena as "sum of propagation phenomena" can be stated, as regards field equations.

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[2] Owing to Clebsch's theorem

In another way, beside the quoted Author, [1], we can explain here such a decomposition taking into account, for example, Eq. 1'.

The vector $\frac{\partial^2 \mathbf{s}_1}{\partial t^2}$ results irrotational like $\mathbf{s}_1$ . In fact:

$$\mathrm{rot}\frac{\partial^2 \mathbf{s}_1}{\partial t^2} = a^2 \mathrm{rot}\Delta \mathbf{s}_1 ,$$

where in the second member one can invert between them rot and Δ, getting:

$$\mathrm{rot}\frac{\partial^2 \mathbf{s}_1}{\partial t^2} = 0 .$$

Thus, the irrotational component $\mathbf{s}_1$ raises irrotational accelerations. These ones, in a time interval *dt*, raise the irrotational speed increases $d\frac{\partial \mathbf{s}_1}{\partial t}$.

Similar considerations, that we leave to the reader, can be made for Eq. 1", for which solenoidal displacements raise accelerations and solenoidal speed increases.

Therefore, among irrotational phenomena and solenoidal ones, there is a separation: the first ones don't generate the other ones. Thus, in a well posed problem where one gives initial values of **s** and of $\frac{\partial \mathbf{s}}{\partial t}$ , both these vectorial fields can be decomposed in irrotational parts and solenoidal ones to make two separated problems, as regards field equations. On the contrary, such two behaviours are not separated because of the boundary conditions, where waves of one kind can raise waves of the other kind.

## 3. Oscillations of Any Kind

Following again Persico, [1], the most general kind of oscillation can be easily led back to a combination of two cases, decomposing the concerned displacement **s** as sum of two displacements $\mathbf{s}_1$ and $\mathbf{s}_2$, an irrotational one and a solenoidal one. And this is always possible: in fact we can always determine such a function $\varphi$ that[3]:

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[3] Look at Eq. 2: it's the well known Poisson's equation and of this one we know the integral:

$$\Delta\varphi = \text{div}\,\mathbf{s}. \tag{2}$$

It follows that:

$$\text{div}\,(\mathbf{s} - \text{grad}\,\varphi) = 0,$$

then we can find such a vector $\Omega$ that:

$$\mathbf{s} - \text{grad}\,\varphi = \text{rot}\,\Omega.$$

Thus, it's sufficient to take

$$\mathbf{s}_1 = \text{grad}\,\varphi \quad \text{and} \quad \mathbf{s}_2 = \text{rot}\,\Omega,$$

because $\mathbf{s}_1$ results irrotational and $\mathbf{s}_2$ is solenoidal. So, one can see that the most general kind of oscillation is given by the superimposition of two kinds of investigated waves. And, usually, except for particular cases, two kind of waves are presented simultaneously (moreover, the reflection of waves of one kind can also create waves of the other kind). In this connection, we have to add that along the surface of the elastic body, only waves concerning the superficial layer can propagate (Rayleigh's waves). They propagate with a different speed from *a* and *b* and are produced in the reflection of the waves of the above investigated kind .

## 4. General Properties of Wave Equation

Always following Persico, [1], the equation that rules elastic oscillations is written in the following shape[4] [1]:

$$\Delta\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}, \tag{3}$$

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$$\varphi = -\frac{1}{4\pi}\int_S \frac{\text{div}_P\mathbf{s}}{r}\,dS.$$

[4] Eq. 3, using the symbol (by D'Alembert):

$$= \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},$$

is often written: $\psi = 0.$

where $\psi$ can be identified as anyone of the displacement components (or as the "potential of displacements" $\varphi$, or as a component of $\Omega$), and $c$ as $a$ or $b$.

Particular solutions represent plane waves or spheric ones and $c$ is the propagation speed. But Eq. 3 allows much more general solutions.

Solutions written in the shape, [1]:

$$\psi = A\,(x, y, z)\,F\,(f\,(x, y, z) - ct) \tag{4}$$

are particularly important and Eq. 4 represents a more general shape of wave propagation. In this one, the mains solutions are included: for $A = 1$, $f = \pm x$, $F$ is arbitrary and one has plane waves, parallel to the plane $(y, z)$; for $A = 1/r$, $f = \pm r$, $F$ is arbitrary and one has spheric waves. The surfaces $f\,(x, y, z)$ = const. are called "wave surfaces" and their normal paths are called "radii". Generally the elastic displacement **s** has an arbitrary trend with respect to the radius: "longitudinal" and "transverse" waves are only particular cases. A general property of every (regular) solution of Eq. 3 is the generalization of the "mean value theorem" for harmonic functions.

Let us consider an arbitrary point P and plot a sphere $\sigma$ with centre P and radius $r$.

The mean value of $\psi$ on surface $\sigma$ at time $t$:

$$\overline{\psi}\,(r, t) = \frac{1}{4\pi r^2} \int_\sigma \psi d\sigma$$

using polar co-ordinates $r, \theta, \varphi$ of pole P, and putting briefly $\sin\theta\, d\theta\, d\varphi = d\omega$ so that $d\sigma = r^2 d\omega$, (then, $d\omega$ represents the element of solid corner) becomes:

$$\overline{\psi}\,(r, t) = \frac{1}{4\pi} \iint \psi d\omega$$

(the double integral must be meant extending from 0 to $2\pi$ for $\varphi$ and from 0 to $\pi$ for $\theta$).

From Eq. 3 we get the differential equation for $\overline{\psi}$, [1]:

$$\frac{\partial^2 (r\overline{\psi})}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2 (r\overline{\psi})}{\partial t^2}.$$

Then, the function ( in $r$ and $t$ ) "$r\overline{\psi}$" satisfies the equation of vibrating string. From this one we can find a notable formula, due to Poisson, that gives the general solution of Eq. 3 (supposed right in an unlimited space), when the initial values $\psi_0$ and $\dot{\psi}_0$ of $\psi$ and of $\dfrac{\partial\psi}{\partial t}$ are given in all the space, [1]:

$$\psi_P(t) = t\dot{\overline{\psi}}_0(ct) + \frac{d}{dt}\left[t\dot{\overline{\psi}}_0(ct)\right], \qquad (5)$$

where the subscript $_{(0)}$ means "at zero time" and the argument "t" in the R.S.M. is a distance r from the origin, divided by c.
The symbol $\dot{\overline{\psi}}$ means:

$$\dot{\overline{\psi}} = \frac{\partial}{\partial t}\overline{\psi} = \frac{1}{4\pi r^2}\frac{\partial}{\partial t}\int_\sigma \psi d\sigma = \frac{1}{4\pi^2 r^2}\int_\sigma \frac{\partial\psi}{\partial t}d\sigma .$$

In order to know the value of $\psi$ in P at a general time $t$, only the mean values, upon a sphere with centre P and radius $ct$, of the given functions $\psi_0$ and $\dot{\psi}_0$ are used. It means that only initial conditions of the medium in points, far $ct$ from P, affect the condition of the medium in P at time $t$. These results of mathematical physics that we have derived from the clear presentation of Persico, [1], will be used here as fundaments for a proposal of discretization. Eq. 5 is here used in attempt to make the time discrete, for which we can consider increases[5] $\Delta t$, getting:

$$\psi(\Delta t) = \Delta t\dot{\overline{\psi}}_0\ (c\Delta t) + \overline{\psi}_0(c\Delta t), \qquad (6)$$

where each step $\Delta t$ of the L.S.M. start from a time t = 0, where $\overline{\psi}_0$ and $\dot{\overline{\psi}}_0$ of the R.S.M. are calculated.

Equation (6) is coherente with the approximation:

$$\frac{d}{dt}\left[t\overline{\psi}_0(ct)\right] \cong \frac{t\overline{\psi}_0(ct)\Big|_0^{\Delta t}}{\Delta\tau}$$

It is possible to introduce other approximations, as, for example:

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[5] When applied to the time, t, Δ means interval.

$$\frac{d}{dt}\left[t\overline{\psi}_0(ct)\right]\cong\frac{t\overline{\psi}_0(ct)\Big|_0^{2\Delta t}}{2\Delta\tau}.$$

giving different algoritms from that indicated in the present paper, whose scope is only to verify the possibility of finite difference algoritms.

In this equation *c* takes value *a* or *b* whenever irrotational phenomenon or solenoidal one are considered. Quantity $c\Delta t$ is a distance. The central idea to make the space discrete, or to single out the "nodes", is represented by making $c\Delta t$ as a distance which allows to go from node to node either with $c=a$ or with $c=b$.

Eq. 5 appears as the foundamental tool for taking into account all the implications of field equation in a way that allows to procede by steps in time. As boundaries are concerned, this work is an attempt to take into account their effects, including them step by step into the field behaviour described by Eq. 5.

## 5. Discretization of the Analysis

The scope of this paragraph and of the following one's, is only to verify the possibility of algoritms of finite difference type, capable of utilize the propagation nature of the dynamic behaviour of continuous homogeneus media. A cubic reticulum (fig. 1) shows, for each examinated point, eight points (making a centred cube) that belong to a sphere with radius equal to every entire multiple of the reticulum semidiagonal. If the radii of the spheres of influence are known, we could find such a reticulum that such two radii result both multiplies (in case, in an approximative way) of the diagonal or, in another approach, adopt the greater of the two radii as diagonal of the cube and interpolate on the diagonals in order to have points whose distance from the centre is equal to the smaller one. In such a way, making every sphere discrete with its eight points, one can try to follow the evolution of each dynamic phenomenon.

At each time, the state must be decomposed in two characteristic propagation components. Then, we calculate the state at a following time (for discrete intervals). The procedure is repeatedly applied to this new state. Let us define the following symbols:

$x,y,z$ = orthogonal cartesian axes,

$P_{i,j,k}$ = points making a cubic reticulum on $x,y,z,$ with cubes with diagonal $d = a\Delta t$ and side $l = a\Delta t/\sqrt{3}$, where i,j,k take entire values ($P_{i,j,k} = P(x_i, y_j, z_k)$),

$f(x,y,z,t)$ = function of space and time,

$f_{i,j,k}$ $(t)$ = value of $f$ $(x,y,z,t)$ in $P_{i,j,k}$ at time $t$, **s** $(x,y,z,t)$ = time function vectorial field, meaning displacements,
$\dot{\mathbf{s}}$ $(x,y,z,t)$ = time function vectorial field, meaning speeds (coupled with **s**),
$u,v,z$ = components of **s**.

$\overline{OA} = \overline{OB} = \overline{OC} = \overline{OD} = \overline{OA'} = \overline{OB'} = \overline{OC'} = \overline{OD'} = a\Delta t$
$\overline{OA^*} = \overline{OB^*} = \overline{OC^*} = \overline{OD^*} = \overline{OA'^*} = \overline{OB'^*} = \overline{OC'^*} = \overline{OD'^*} = b\Delta t$
$O \equiv P\ (x_i\ , y_j\ , z_k)$

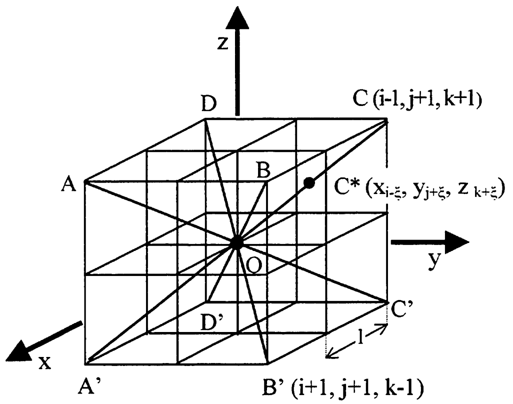



*Figure 1*

Let us write the sequence (order is not important) of points $P(x_{i\pm\xi}, y_{j\pm\xi}, z_{k\pm\xi})$, eight at all, with $\boldsymbol{m}_{i,j,k}$, where signs ± of sub index are taken only once, for fixed i,j,k. Moreover, $\xi = 1$ for irrotational phenomena and $\xi = b/a$ for the solenoidal ones. Let us write with $\boldsymbol{n}_{i,j,k}$ the set of points $P_{i^*,j^*,k^*}$, where $i^*,j^*,k^* \neq i,j,k$, and $i^* = i \pm (2\alpha+1)$, $j^* = j \pm (2\beta+1)$, $k^* = k \pm (2\gamma+1)$ with $\alpha, \beta, \gamma = 0, 1, 2$ ...

*Table 1*- Discretization of divergence calculation

| Differential | Discretized |
|---|---|
| $\text{div}\,\mathbf{s} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$ | $(\text{div}\,\mathbf{s})_{i,j,k} = \dfrac{u_{i+1,j,k} - u_{i-1,j,k}}{2l} + \dfrac{v_{i,j+1,k} - v_{i,j-1,k}}{2l} + \dfrac{w_{i,j,k+1} - w_{i,j,k-1}}{2l}$ |

*Table 2*- Discretization of Poisson's integral

| Differential | Discretized |
|---|---|
| $\varphi = -\dfrac{1}{4\pi}\int_S \dfrac{\text{div}_P\,\mathbf{s}}{r} dS$ | $\varphi_{i,j,k} = -\dfrac{1}{4\pi}\sum_n \dfrac{(\text{div}\mathbf{s})_{i^*,j^*,k^*}}{r_{i,j,k,i^*,j^*,k^*}} 8l^3$ |

*Table 3*- Discretization of calculation of irrotational component

| Differential | Discretized |
|---|---|
| $s_1 = \text{grad}\ \varphi$ | $u_{1\ i,j,k} = \dfrac{\varphi_{i+1,j,k} - \varphi_{i-1,j,k}}{2l}$ <br> $v_{1\ i,j,k} = \dfrac{\varphi_{i,j+1,k} - \varphi_{i,j-1,k}}{2l}$ <br> $w_{1\ i,j,k} = \dfrac{\varphi_{i,j,k+1} - \varphi_{i,j,k-1}}{2l}$ |

*Table 4*- Discretization of calculation of solenoidal component

| Differential | Discretized |
|---|---|
| $s_2 = s - \text{grad}\ \varphi$ | $u_{2\ i,j,k} = u_{i,j,k} - u_{1\ i,j,k}$ <br> $v_{2\ i,j,k} = v_{i,j,k} - v_{1\ i,j,k}$ <br> $w_{2\ i,j,k} = w_{i,j,k} - w_{1\ i,j,k}$ |

*Table 5*- Discretization of derivation

| Differential | Discretized |
|---|---|
| $\dfrac{\partial f}{\partial t} = \dot{f}$ | $t_{\tau+1} = t_\tau + \Delta t$ <br> $\dot{f}_{i,j,k}(t_{\tau+1}) = \dot{f}_{i,j,k}(t_{\tau^*}) + 2\dfrac{f_{t_{\tau+1}} - f_{t_{\tau^*}}}{\Delta t}$ <br> $(\tau = 0, 1, 2, \ldots=)$ |

*Table 6*- Discretization of propagation effect

| Differential | Discretized |
|---|---|
| $\psi(t_1+\Delta t) = 2\Delta t\dot{\overline{\psi}}_0(c\Delta t)+$ $+\overline{\psi}_0(c\Delta t)$ | $\psi_{i,j,k}(+\Delta t)=\Delta t\sum_{m}\frac{\pi}{2}\times\times$ $(c\Delta t)^2\dot{\psi}(x_{i\pm\xi}, y_{j\pm\xi}, z_{k\pm\xi})+$ $+\sum_{m}\frac{\pi}{2}(c\Delta t)^2\psi(x_{i\pm\xi}, y_{j\pm\xi}, z_{k\pm\xi})$ where $f(x_{i\pm\xi}, y_{j\pm\xi}, z_{k\pm\xi}) = f_{i,j,k}+$ $+\xi(f_{i\pm1,j\pm1,k\pm1} - f_{i,j,k})$ |

## 6. Initial Condition

One takes an arbitrary initial condition $t_0 = t(0) = 0$ in the shape of arbitrary values $u_{i,j,k}$, $v_{i,j,k}$, $w_{i,j,k}$, provided that the boundary conditions are satisfied, and (for example) of null values for $\dot{u}_{i,j,k}$, $\dot{v}_{i,j,k}$, $\dot{w}_{i,j,k}$.

## 7. Sequence of Calculation Operations

1) With each component of the vector:

$$\begin{cases} u_{i,j,k}(t) \\ v_{i,j,k}(t) \\ w_{i,j,k}(t) \end{cases}$$

where $t = 0, \Delta t, 2\Delta t, ...$, one calculates $(\text{div } \mathbf{s})_{i,j,k}$, with the formula of Table 1.
2) Using the Poisson's integral discretization in Table 2, one calculates $\varphi_{i,j,k}(t)$.
3) One calculates the irrotational component $s_{1_{i,j,k}}$ by means of its three components:

$$\begin{cases} u_{1_{i,j,k}} \\ v_{1_{i,j,k}} \\ w_{1_{i,j,k}} \end{cases}$$

using table 3.

4) One calculates the solenoidal component $\mathbf{s}_{2_{i,j,k}}$, by means of its three components:

$$\begin{cases} u_{2_{i,j,k}} \\ v_{2_{i,j,k}} \\ w_{2_{i,j,k}} \end{cases}$$

using table 4.

5) One calculates the derivative with time of every component $u_i(t)$, $v_i(t)$, $w_i(t)$ (i = 1, 2), generally indicated with $f(t)$, using the values in $t$ and $t - \Delta t$, according with table 5.

6) One calculates the value of each component $u_i(t)$, $v_i(t)$, $w_i(t)$, at time $t + \Delta t$, with table 6, where $\boldsymbol{m}$ is the sequence of the nodes placed on the sphere with radius $a$ and $b$, respectively for i = 1, i = ξ. The derivatives (predetermined) are necessary.

7) One makes the sum of irrotational and solenoidal components getting the components of:

$$\mathbf{s}_{\text{internal}}(t + \Delta t)$$

in all the points which don't belong to the boundary. On the boundary, one adopts the appropriate component values:

$$\begin{cases} u_{\text{boundary}}(t + \Delta t) \\ v_{\text{boundary}}(t + \Delta t) \\ w_{\text{boundary}}(t + \Delta t) \end{cases}$$

8) One applies again all the process to the obtained vector $\mathbf{s}\ (t + \Delta t)$, for a further step $\Delta t$.

## 8. Boundary Conditions

A finite difference method makes us free from the necessity of obtaining an analytical solution of the reflection of waves on the boundary (in particular on a boundary having time variable location). In fact, it is necessary and sufficient that the boundary is composed by points of the network. At various times such points can be different and realize at each time the discretization of a boundary surface. On the contrary, we have the problem of finding out algorithms capables of talking into account boundary effects at a discretized level. In order to do that, let us consider that in a linear analysis, the superimposition of effects is valid. Therefore, we can add the propagations in a field free from external actions and the effect of boundary actions, if it is done in a proper manner.

Eq. 5 holds in the case where the medium is extended far to infinite. On the other hand, the integral determining $\Delta\varphi$ in Eq. 2 must be extended to every elementar volume in which div $\mathbf{s} \neq 0$. Let us consider a problem where (fig. 2) points like A, B, C, D are on the boundaries. The points indicated with a litte cross, like M' and N', on the contrary, are a virtual continuation of the definition field.

In the case of a restraint stopping movements of points like C, nothing forbids us to consider even points like M' and N'. Thus, it will be sufficient to calculate at each step the displacements of points like M, taking into account the propagation from B, C, R, S. The divergence in C can be obtained from the displacements in C' (calculated) and C'' (null) and from those ones of other points of the boundaries (null).

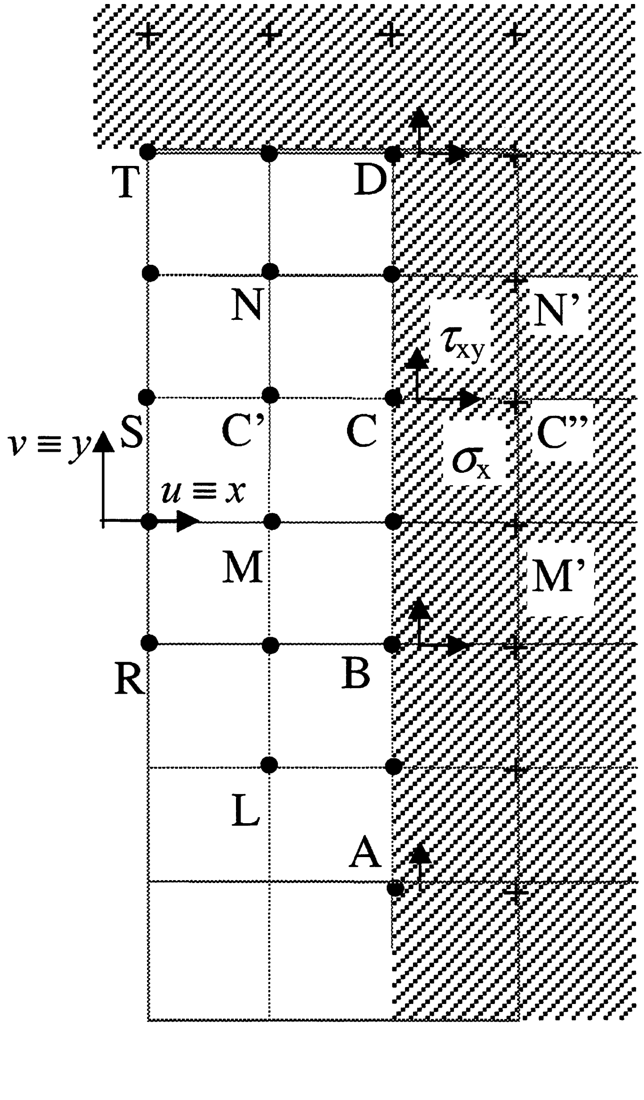
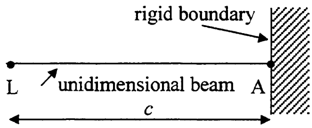
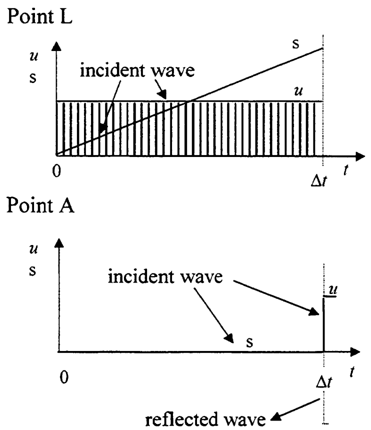


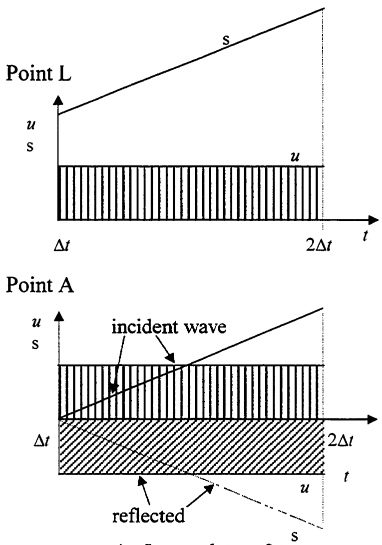

*Figure 2* - Boundary condition (the scheme refers to a bidimensional problem)




a) - Scheme of the beam




b) - Interval 0 ÷ Δt




c) - Interval Δt ÷ 2Δt

*Figure 3* - Unidimensional beam with axial propagation

The divergence in every "virtual" point is obviuosly null.

In the case of boundary with known external stresses (in particular null ones), in addition to the displacements of points like C we need to calculate also those ones of points like M'. The displacements of C are obtained for propagation from those of M, N, M' and N'. The displacements of M' are obtained from boundary conditions, known those of M, C, and B, supposing, at the end of interval $\Delta t$, an external action realizing the boundary conditions. The same external action realizes a zero value of divs in every "virtual" point.

Let us consider now the case of (rigid) boundary conditions $\mathbf{s}(t) = 0$ at points as A, B, C, D (i.e. also $\dot{\mathbf{s}} = 0$). The displacement **s** at points as L, M, N at time $\Delta t$ can be obtained as previously described.Points A, B, C, D should have displacements in $\Delta t$ if considered free, and the boundary condition effect is equivalent to external actions at $\Delta t$ on the dashed part, which realize for them $\mathbf{s} = 0$. Such external actions can't have effects on L, M, N at time $\Delta t$. At the beginning of a further $\Delta t$, we can consider all the points (including A, B, C, D) free from external actions. This implies the applicability of the theory of wave propagations. We can try a clarification referring to the case of unidimensional longitudinal waves (see fig. 3) of a constant stress and of speed $c$, reflected by a rigid boundary ($\Delta t = l/c$). At time $2\Delta t$ in L there are no effect of tipe s due to the application in A of a reflected wave at time $\Delta t$. As $2\Delta t \div 3\Delta t$ and subsequent intervals are concerned, the effect of the external actions (reflected wave) are completely described in L by the reflected wave itself. Let us come back to fig. 2. The suggested procedure allows us to take easily into account the effects of time variable location of the restraints. It is sufficient to impose that the conditon $\mathbf{s} = 0$, applied in points like A, B, C, D, during a convenient discrete time interval ($n\Delta t$, with $n$ an integer), is applied in points like P, Q, R, during another convenient interval. Obviously $n$ must be related to the speed of the boundary conditions in the continuum. The case of boundary conditions imposing particular stress values at the boundary (free boundary or known external actions) can be approached as follows. Let us suppose that the boundary is a plane perpendicular to $x$, where $\sigma_x$ , $\tau_{xy}$ and $\tau_{xz}$ are known at time $\Delta t$ in A, B, C and D, (fig. 2).

Displacements $\mathbf{s}_{C'}$ , $\mathbf{s}_{D'}$ are calculated by means of the proposed procedure. It is sufficient to pose:

$$u_{D''} = u_{D'} + \frac{1}{\lambda+\mu}\ \sigma_{D,x}\, 2l \quad ,$$

$$u_{C''} = u_{C'} + \frac{1}{\lambda+\mu}\ \sigma_{C,x}\, 2l \quad ,$$

$$u_{B''} = u_{B'} + \frac{1}{\lambda+\mu}\ \sigma_{B,x}\, 2l \ ,$$

$$v_{C''} = v_{C'} + 2l\left(\frac{\tau_{x,y}}{\mu} - \frac{u_N - u_M}{2l}\right), \ ....$$

and so on.

It is beyond the scope of this paper to discuss the case of boundary conditions imposing particular stress values at the boundary (free boundary or known external actions).

Such a case is complicated by the fact that the utilized propagation refers to displacements. For the determination of the influence in a point P at a time $t + \Delta t$ of the conditions at the time $t$ on the sphere of radius $v\Delta t$ (where $v$ is the propagation speed of the kind of the considered waves), when P is near the boundary correspondent to the time $t$, it is sufficient to take into account an influencing sphere having points out of the boundary and at rest.

## 9. Conclusions

The properties of linear continuous dynamics as wave propagation have been recalled, in particular a theory that permits the determination of the behaviour in a point at a time from the behaviour on a concentric sphere at a previous time. Such theory need the separation of the displacement, at each time, into its irrotational and solenoidal part. A discretization by means of finite differences is proposed. Such a discretization allows to determine the evolution of dynamic phenomena, including the effect of boundary conditions regarding the displacement, also in the case of boundary location depending on time. The proposed discretization is a practical demonstration of the possibility of a finite difference approach, even in the case of time dependent boundary location. Ameliorations and reiterations are obviously possible or necessary in order to obtain in calculations reliable results.

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